# Wideband Analysis of Scattering Problems Using an Eigenmode Projection Technique

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***Abstract*—Wideband analysis of dielectric objects illuminated by uniform plane waves is performed using an eigenmode projection technique. The frequency independent feature of the generated matrices of the eigenmode projection method will be exploited to solve electromagnetic scattering problems over a wide range of frequencies efficiently without the need of filling and inverting all the system of matrices and the encountered numerical integrations are only evaluated once, with their values used at all frequencies. Results are presented to validate the method and illustrating the speed up of the technique making use of the frequency independent feature of the proposed method.**

## I. Introduction

Wideband analysis in electromagnetics is of great importance for a wide range of applications and has thus been investigated using different time and frequency domain methods. In frequency domain methods such as the moment method (MoM), the solution procedure requires filling and inverting the matrix at each frequency point, which is a prohibitively time and memory consuming process.

An eigenmode projection technique was proposed to solve problems of microwave cavities [1] and free-space scattering from dielectric objects [2]. In this work, the frequency independent feature of the generated matrices of the eigenmode projection method will be exploited to solve electromagnetic scattering problems over a wide range of frequencies efficiently. A general overview of the eigenmode projection method is given in the next section, followed by the procedure employed to produce the wideband solution without the need of filling and inverting all the system of matrices and the encountered numerical integrations are only evaluated once, with their values used at all frequencies. Results are presented to validate the method, illustrating the speed up of the technique because of the frequency independent feature of the proposed method.

## II. Overview of the Eigenmode Projection Technique

An arbitrary dielectric object is excited by uniform plane wave as shown in Fig. 1. A fictitious canonical cavity with perfect magnetic (PM) or perfect electric (PE) boundary is chosen to provide the set of eigenmodes that is used in solving the problem. The cavity is chosen to be either cylindrical or spherical according to the problem under consideration either being two dimensional (2D) or three dimensional (3D), respectively.

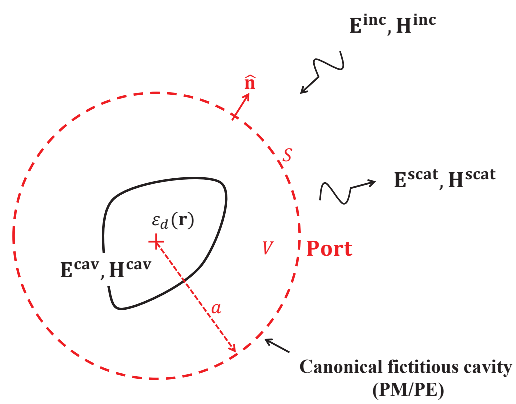


Fig. 1. General scattering problem by an arbitrary dielectric object with permittivity function $\varepsilon_d(\mathbf{r})$, the object is enclosed by a canonical fictitious cavity with radius $a$ , volume $V$ and outer surface $S$.

The total port field is the summation of the incident field and the scattered port modes as:

$$\mathbf{E}^{\mathbf{port}}(\mathbf{r}) = \mathbf{E}^{\mathbf{inc}}(\mathbf{r}) + \mathbf{E}^{\mathbf{scat}}(\mathbf{r}) \tag{1}$$

$$\mathbf{H}^{\mathbf{port}}(\mathbf{r}) = -\frac{1}{j\omega\mu_0}\nabla \times \mathbf{E}^{\mathbf{port}}(\mathbf{r}) \tag{2}$$

where $\mathbf{E}^{\mathbf{scat}}(\mathbf{r}) = \sum_p a_p^{scat}\mathbf{D}_p(\mathbf{r})$, with $\mathbf{D}_p(\mathbf{r})$ being the functional dependence of the scattered fields from cylindrical or spherical objects in case of 2D or 3D problems, respectively [3]. Further illustration is available in the Appendix.

Cavity eigenmodes are a set of orthogonal solenoidal ($\mathbf{E}_n$, $\mathbf{H}_n$), divergence free, and irrotational ($\mathbf{F}_\alpha$, $\mathbf{G}_\lambda$), curl free, modes. The solenoidal modes are related together by $\nabla \times \mathbf{E}_n = k_n\mathbf{H}_n$ and $\nabla \times \mathbf{H}_n = k_n\mathbf{E}_n$, however the irrotational modes are represented by the divergence of scalar potentials $l_\alpha\mathbf{F}_\alpha = \nabla\phi_\alpha$ and $l_\lambda\mathbf{G}_\lambda = \nabla\psi_\lambda$, where $k_n$, $l_\alpha$, $l_\lambda$ are wave numbers for solenoidal, irrotational electric and magnetic fields respectively [4], [5]. Thus the cavity fields could be expressed as follows:

$$\mathbf{E^{cav}}(\mathbf{r}) = \sum_n a_n \mathbf{E}_n(\mathbf{r}) + \sum_\alpha f_\alpha \mathbf{F}_\alpha(\mathbf{r}) \quad (3)$$

$$\mathbf{H^{cav}}(\mathbf{r}) = \sum_n b_n \mathbf{H}_n(\mathbf{r}) + \sum_\lambda g_\lambda \mathbf{G}_\lambda(\mathbf{r}) \quad (4)$$

The canonical cavity is chosen with PM boundary to eliminate the irrotational magnetic field [1]. Applying fields expansions to Maxwell's equations and performing modes projection leads after some lengthy algebraic manipulations to the following equations

$$k_n a_n = -j\omega\mu_0 b_n \quad (5)$$

$$k_n b_n + \oint_S \left(\mathbf{H^{port}}(a) \times \mathbf{E}_n(a)\right) \cdot ds = j\omega \left[\sum_m \langle \mathbf{E}_n,\, \mathbf{E}_{n'} \rangle\, a_n + \sum_{\alpha'} \langle \mathbf{E}_n,\, \mathbf{F}_\alpha \rangle\, f_\alpha \right] \quad (6)$$

$$\oint_S \left(\varepsilon_0 \mathbf{E^{port}}(a) \phi_\alpha(a)\right) \cdot ds - l_\alpha \left[\sum_m \langle \mathbf{F}_\alpha,\, \mathbf{E}_n \rangle\, a_n + \sum_{\alpha'} \langle \mathbf{F}_\alpha,\, \mathbf{F}_{\alpha'} \rangle\, f_\alpha \right] = 0 \quad (7)$$

where, $\langle \mathbf{X},\, \mathbf{Y} \rangle = \int_V \varepsilon(\mathbf{r}) \mathbf{X}(\mathbf{r}) \cdot \mathbf{Y}(\mathbf{r}) dv$.

Combining the equations (5-7) yields a matrix equation of the form:

$$\left[[EE_{nn'}] - [EF_{n\alpha}]\,[FF_{\alpha\alpha'}]^{-1}\,[EF_{n\alpha}]^T\right][a_n] - \frac{k_n^2}{\omega^2\mu_0}[a_n] + \frac{1}{l_\alpha}[EF_{n\alpha}]\,[FF_{\alpha\alpha'}]^{-1} \oint_S \left(\varepsilon_0 \mathbf{E^{port}}(a)\phi_\alpha(a)\right) \cdot ds - \oint_S \left(\mathbf{H^{port}}(a) \times \mathbf{E}_n(a)\right) \cdot ds = 0 \quad (8)$$

where $[EE_{nn'}]$, $[EF_{n\alpha}]$ and $[FF_{\alpha\alpha'}]$ are matrices with their elements are the cavity eigenmode projections $\langle \mathbf{E}_n,\, \mathbf{E}_{n'} \rangle$, $\langle \mathbf{E}_n,\, \mathbf{F}_\alpha \rangle$ and $\langle \mathbf{F}_\alpha,\, \mathbf{F}_{\alpha'} \rangle$ respectively and $n$, $n'$, $\alpha$ and $\alpha' = 1, 2, \cdots N$ with $N$ being the number of cavity modes under consideration.

It should be noted that the cavity eigenmodes are frequency independent and the frequency dependence in the equations appears only in the frequency $\omega$ and the port fields $\mathbf{E^{port}}$ and $\mathbf{H^{port}}$ which include implicitly the scattered field coefficients $a_p^{scat}$ , thus there exists two unknowns in (8) which are $[a_n]$ and $\left[a_p^{scat}\right]$

In order to evaluate the fields coefficients another equation is obtained by enforcing the boundary conditions by equating the tangential electric fields components for port and cavity modes at cavity boundary, this is done in average sense by performing surface projecting over $\mathbf{D}_{p'}$ to avoid discontinuity at the surface as follows:

$$\oint_S \hat{\mathbf{n}} \times \mathbf{E^{port}}(a).\mathbf{D}_{p'}(a) ds = \oint_S \hat{\mathbf{n}} \times (\sum_n a_n \mathbf{E}_n(a) + \sum_\alpha f_\alpha \mathbf{F}_\alpha(a)).\mathbf{D}_{p'}(a) ds \quad (9)$$

Equations (1-2), (8) and (9) can be manipulated and cast in the matrix form:

$$[\mathbf{A} + \mathbf{B}(\omega)]\,[a_n] = \mathbf{C}(\omega),$$
$$\mathbf{A} = [EE_{nn'}] - [EF_{n\alpha}]\,[FF_{\alpha\alpha'}]^{-1}\,[EF_{n\alpha}]^T \quad (10)$$

It is obvious that all the elements of $\mathbf{A}$ are integrations of the cavity mode projections and is thus frequency independent, whereas the elements of $\mathbf{B}(\omega)$ and $\mathbf{C}(\omega)$ contain integrations over the fictitious port and is thus frequency dependent due to the argument $(ka)$ of the used Hankel functions in $\mathbf{D}p$ with $k$ is the propagation constant and $a$ is the fictitious cavity radius.

## III. Wideband Analysis

The solution of (10) requires the evaluation of the inverse $[\mathbf{A} + \mathbf{B}(\omega)]^{-1}$, which is evaluated in light of the theorem in [6] as follows:

let $\mathbf{C}_1 = \mathbf{A}^{-1}$

$$, \mathbf{C}_i = \mathbf{C}_{i-1} - \mathbf{g}_{i-1}\mathbf{C}_{i-1}\mathbf{B}_{i-1}\mathbf{C}_{i-1} \quad (11)$$

, $\mathbf{B}_i$ is generated by setting all rows in $\mathbf{B}$ to zero except the $i^{th}$ row, and $g_i = 1/\left(1 + trace(\mathbf{C}_i\mathbf{B}_i)\right)$, with $i = 2, \ldots, N$ and $N$ is the matrix dimension which is the number of eigenmodes under consideration, then the required inverse may be obtained using:

$$[\mathbf{A} + \mathbf{B}(\omega)]^{-1} = \mathbf{C}_N - \mathbf{g}_N\mathbf{C}_N\mathbf{B}_N\mathbf{C}_N \quad (12)$$

In the previous inversion scheme it should be mentioned that the evaluation of $\mathbf{A}^{-1}$ is done only once at some reference frequency and is stored, for each frequency applying the previous procedure the inverse $[\mathbf{A} + \mathbf{B}(\omega)]^{-1}$ is obtained with no further inversion needed. The inversion scheme is performed in forward not recursive manner avoiding memory overloading resulting from recursion.

By this the time consuming process of matrix inversion is performed only once unlike other frequency domain methods.

The second interesting thing about the proposed technique is the time saving in the matrix filling process for wideband analysis. By inspecting the matrices in (10) it is found that the matrix $\mathbf{A}$ as mentioned before is frequency independent and thus it is evaluated only once, while matrices $\mathbf{B}(\omega)$ and $\mathbf{C}(\omega)$ are partially evaluated at the different frequency points for only the terms with the surface integral of the port modes which end up in a closed analytical form where some sorts of orthogonality are used converting the surface integrals to direct substitution in the fields expression while the terms with the volume integrals of the eigenmodes is frequency independent and evaluated only once, as a result the filling process is done in a fast manner without the need of filling all the system of matrices at each frequency step for wideband analysis.

## IV. Results and Verification

Bistatic scattering width (SW), normalized to free space wavelength ($\lambda$), [3] of 2D dielectric objects illuminated by transverse magnetic ($\mathrm{TM}^z$) plane wave is used to verify the proposed technique. The number of eigenmodes required for the solutions was found to be an increasing function of frequency and is taken by the rule of thumb $N = \mathrm{ceil}(8k_d a)$ with $k_d$ is the propagation constant in the dielectric material at the working frequency, this rule of thumb showed good agreement with different geometries.

In the proposed approach making use of the frequency independent feature of the generated matrices and filling it only once the number of modes is taken according to the maximum frequency as in Fig. 2 where the results obtained for cylindrical scatterer are compared with the analytical solution in [3] with the number of eigenmodes are set to $\mathrm{ceil}(8k_{d,max}a)$ .

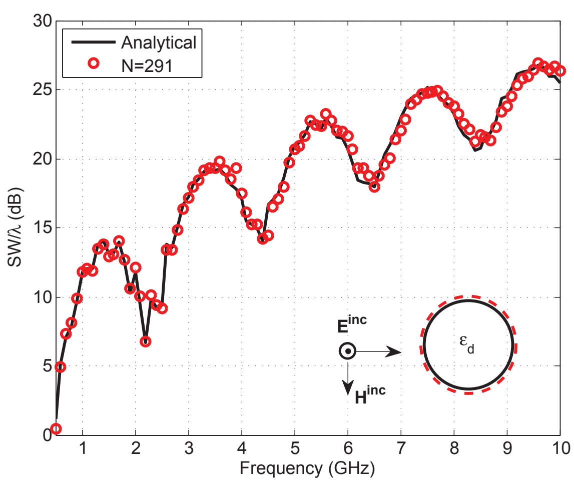


Fig. 2. SW for cylindrical dielectric scatterer with $\varepsilon_d = 3\varepsilon_0$ and radius = 0.1 m illuminated by $\mathrm{TM}^z$ plane wave with canonical cavity touching the dielectric surface compared with the analytical solution.

However solving in direct manner less number of modes are taken for small frequencies but with all the system of matrices calculated at each frequency step. Evaluating the effectiveness of the proposed accelerated approach a speed up factor is introduced to study its advantage over direct solution as follows:

$$\text{Speed up factor} = \frac{\text{Time for direct approach}}{\text{Time for accelerated approach}} \quad (13)$$

Figure 3 shows the speed up factor for the cylindrical scatterer studied in Fig. 2. The number of modes utilized in the direct approach is $\mathrm{ceil}(8k_d a)$ for each frequency points with all the matrices calculated, while the accelerated technique utilize $\mathrm{ceil}(8k_{d,max}a)$ for all frequency points with the frequency independent matrices calculated only once.

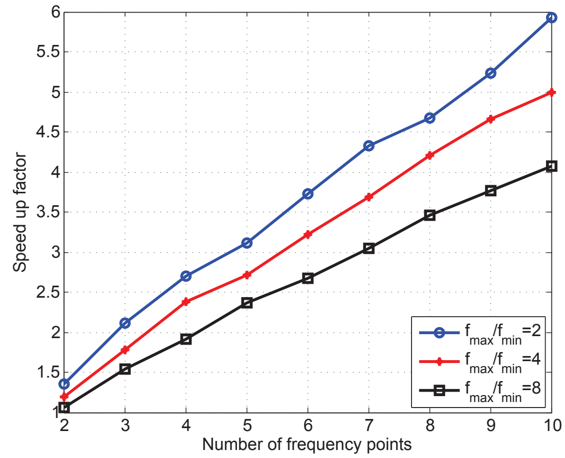


Fig. 3. Speed up factor for cylindrical dielectric scatterer with $\varepsilon_d = 3\varepsilon_0$ and radius = 0.1 m illuminated by $\mathrm{TM}^z$ plane wave with canonical cavity touching the dielectric surface at different maximum-to-minimum frequency ratios, $f_{min} = 1$ GHz.

In Fig. 3 it is worth mentioning that the effectiveness of the proposed accelerated approach, represented by the speed up factor, increases as the number of frequency points increases due to the reduction in the matrices filling time in this approach, even with the direct approach using less number of modes for lower frequencies, however this fact results in decreasing the speed up factor for wider band solutions.

Figure 4 produces another verification result illustrating that the proposed technique is valid for any dielectric geometry compared with the results obtained using the method of moments (MOM).

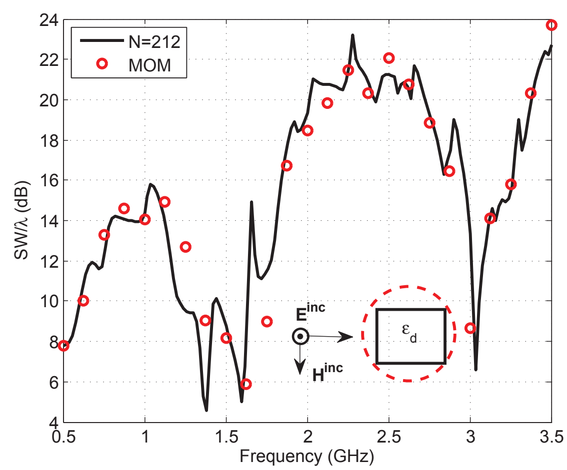


Fig. 4. SW for rectangular dielectric scatterer with $\varepsilon_d = 4\varepsilon_0$ and dimensions $0.2 \times 0.3\,\mathrm{m}^2$ illuminated by $\mathrm{TM}^z$ plane wave incident perpendicular to its short side with canonical cavity passing through the rectangle corners compared with the results obtained using MOM.

Figure 5 shows the result for a coated dielectric cylinder compared with the analytical solution in [3] with the number of eigenmodes is set to $\mathrm{ceil}(8k_{d,max}a)$, where $k_{d,max}$ =

$\frac{\omega_{max}}{C}\sqrt{\varepsilon_{average}}$ and $\varepsilon_{average}$ is given by:

$$\varepsilon_{average} = \frac{\int_{S_d} \varepsilon_d(\mathbf{r})dS}{S_d}, \tag{14}$$

where $S_d$ is the dielectric object surface area.

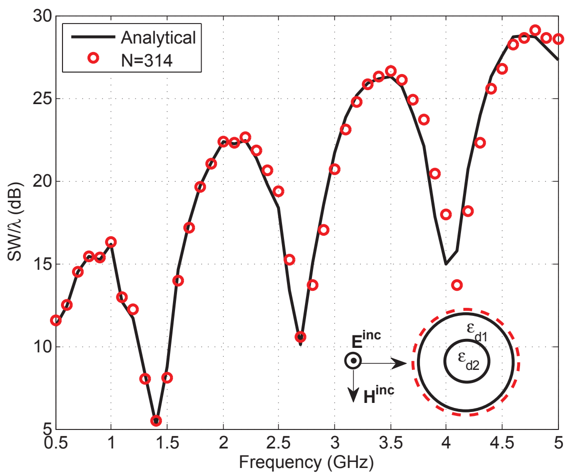


Fig. 5. SW for cylindrical coated dielectric scatterer with $\varepsilon_{d1} = 3\varepsilon_0$, $\varepsilon_{d2} = 2\varepsilon_0$, inner radius of $0.1$ m and outer radius of $0.2$ m illuminated by $\mathrm{TM}^z$ plane wave with canonical cavity touching the outer dielectric surface compared with the analytical solution.

## V. Conclusion

Adopting eigenmode projection technique in electromagnetic scattering problems provides a frequency domain method with a very unique property of the independent feature of the generated matrices. This is useful in wideband analysis which is of great importance for a wide range of applications, where the frequency independent feature is used in solving electromagnetic scattering problems over a wide range of frequencies efficiently without the need of filling and inverting all the system of matrices except at some reference frequency within the band of interest. Results are presented to validate the method and illustrating the speed up of the proposed technique making use of its properties.

## Appendix

In the proposed technique the scattered fields are expanded in terms of cylindrical or spherical functions in case of 2D or 3D problems, respectively following the same procedure described in [3].

For 2D problems the scattered fields are represented by the summation of Hankel functions of the second kind and azimuthal variations. This could be expressed as follows for scattering of $\mathrm{TM}^z$ plane waves:

$$\mathbf{E^{scat}}(\mathbf{r}) = \sum_{p=-\infty}^{\infty} a_p^{scat}\mathbf{D}_p(\mathbf{r}), \tag{15}$$

where $\mathbf{D}_p(\mathbf{r}) = H_p^{(2)}(k\rho)e^{jp\phi}$

However for scattering of $\mathrm{TE}^z$ plane waves it is most convenient to expand the scattered magnetic field rather than the electric field.

Also it is very useful to make use of the cylindrical wave transformations to represent the incident plane wave as in (16), this simplifies the surface integrals making use of the orthogonality between fields.

$$E_0 e^{-jkx} = E_0 e^{-jk\rho cos(\phi)} = E_0 \sum_{p=-\infty}^{\infty} j^p J_p(k\rho)e^{jp\phi} \tag{16}$$

For 3D problems the expansion is done for the magnetic and electric vector potentials $\mathbf{A}(\mathbf{r})$ and $\mathbf{F}(\mathbf{r})$, respectively and they are represented by the summation of spherical Hankel functions of the second kind for radial variations, the exponential functions for $\phi$ variations, and the Legendre polynomials and associated Legendre functions for $\theta$ variations, also the plane waves could be expressed in terms of spherical wave functions. Details for the 3D scattering expansions could be find in [3].

## References


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